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Distribution of Points on a Sphere with Application to Star Catalogs

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A new framework is introduced to analyze the spatial distribution of stars in a catalog, namely, the Voronoi diagram/Delaunay triangulation. The Voronoi diagram is a subdivision of the celestial sphere into polygonal cells, one for each star, so that the cell for star *P* consists of the region on the sphere closer to *P* than to any other star. The Delaunay triangulation is the topological dual with the important property that each spherical cap circumscribing a triangle contains no stars in its interior. Measures of the uniformity in star density and geometric arrangement based on the Voronoi diagram/Delaunay triangulation are presented and compared with existing measures. Methods to generate uniformly distributed points on the sphere, which serve as useful test cases for stellar attitude determination analysis, are formulated and compared. One such method, based on a spherical spiral, is easy to implement and yields a very uniform distribution of points. Finally, the Voronoi density reduction method is introduced to select stars for an onboard catalog from a larger candidate set. The candidate with the smallest Voronoi cell is removed, and the Voronoi diagram of the reduced set is constructed. This process is repeated until the desired number of stars remains.

Introduction

A COMMON problem in spacecraft attitude determination is to select stars for an onboard catalog from a larger set of candidates with the goal that the spatial distribution of the selected stars be uniform. This problem is motivated by the conflicting goals to minimize onboard computer memory requirements and to maximize the probability of having at least one catalog star in a star sensor field of view at any given moment. Several algorithms have been described in the literature that strike a balance between these two goals. Vedder[1] divides the celestial sphere into a number of equal area and contiguous cells and then selects just one star per cell. Barry et al.[2] define a proximity measure for each star to be the sum of the angular distances to the three nearest neighbors. The star with the smallest proximity measure is removed, and the measures for the neighboring stars are recomputed. This process is repeated until the desired number of stars remains. Kudva[3] establishes a set of uniformly distributed reference points on the celestial sphere. The star closest to each reference point is selected for the onboard catalog.

This paper introduces the Voronoi density reduction method to select stars for an onboard catalog. The method is based on the Voronoi diagram[4] of the candidates, which is a subdivision of the celestial sphere into polygonal cells, one for each star, such that the cell for star P consists of the region on the sphere closer to P than to any other star. Figure 1 shows a fragment of a Voronoi diagram (solid line segments) for randomly distributed points. The graph of nearest neighbors (dashed line segments) is the Delaunay triangulation, which is the topological dual of the Voronoi diagram. (Two subdivisions are duals if for each face in the one subdivision there is a corresponding vertex in the other, and for each edge separating two faces in the one subdivision there is an edge connecting the corresponding vertices in the other.) The lines extending through cell edges in the Voronoi diagram are the perpendicular bisectors of the line segments connecting the nearest neighbors.

To select stars for an onboard catalog, the candidate star with the smallest Voronoi cell is removed, and the Voronoi diagram of the reduced set is constructed. This process is repeated until the desired number of stars remains. The method is applied to three examples in the sequel and shown in each case to yield a more uniformly distributed set of stars than the previous methods.

Of course, to evaluate the uniformity of star catalogs it is necessary to have one or more measures of uniformity. In the fields of physics and chemistry, measures of uniformity are based on the potential energy of particles obeying mutual force interaction.[5,6] Of particular importance is the potential law $V(r) = r^{\alpha}$, $-2 < \alpha < 2$, $\alpha \neq 0$ and $V(r) = \ln(1/r)$ for $\alpha = 0$, where r is the Euclidean distance between two particles. The total potential of a set of particles on the sphere is taken to be the measure of uniformity. See Ref. 6 for an excellent discussion and bibliography on these interesting measures of uniformity. There are, however, more appropriate measures for stellar attitude determination.

Vedder[1] defines an R-measure that represents the variation in star density over the celestial sphere. A comparable measure introduced here, the D measure, is based on the Voronoi diagram for points on a sphere. The inverse of the Voronoi cell area, called the Voronoi density, may be interpreted as the star density within the respective cell. The D measure is defined as the variance of the Voronoi density times a normalizing scale factor. The D measure is superior to Vedder's R measure in two ways. First, the R measure will always indicate some nonuniformity in a distribution, even if by all other accounts it is highly uniform. Second, the D measure is intrinsic, whereas the R measure depends on an arbitrary parameter.

A second measure of uniformity defined here is the G measure. It is based on the Delaunay triangulation that possesses the important property that each spherical cap circumscribing a triangle in the triangulation contains no stars in its interior.[4] Applied to stars in a star catalog, the Delaunay caps represent the empty regions between stars. The Delaunay cap areas (DCAs) are summed, and a normalizing function is applied to yield the G measure. The G measure provides a statistical measure of the geometrical arrangement of the stars, namely, how close the triangles in the Delaunay triangulation are to being equilateral.

A final topic covered in this paper is the generation of uniformly spaced points on the sphere. To evaluate star identification schemes, it is desirable to generate a large set of uniformly distributed sample star sensor bore sights. It is relatively easy to generate a uniformly random set of points on the sphere,[7] but such a set will have regions of higher and lower than average density, as chance may have it. A method sometimes used for finite element analysis places points at regular intervals on the faces of an icosahedron.[8] The points are then projected radially onto the sphere. This method has been used recently by Kudva[3] and Ketchum and Tolson[9] in the field of spacecraft attitude determination. Whereas this distribution of points is much more uniform than randomly distributed points, the density of points near the vertices of the icosahedron is still about twice that of points near the centers of the faces. Several refinements of this method

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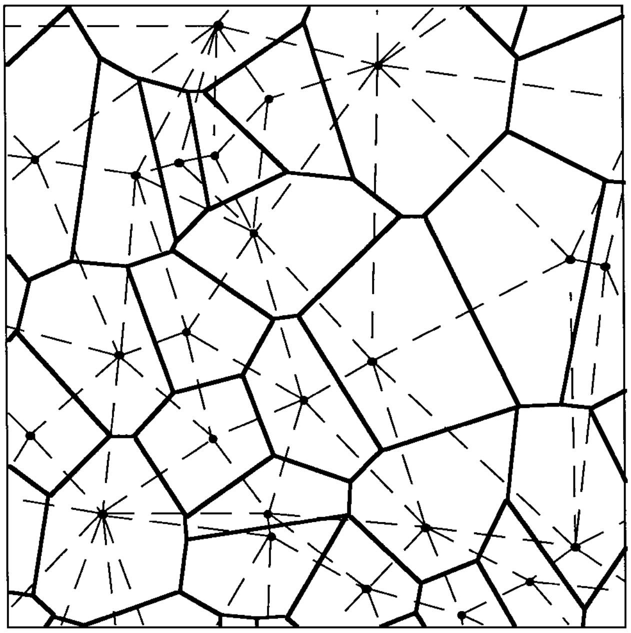


Fig. 1 Voronoi diagram/Delaunay triangulation for points in the Euclidean plane.

have been considered in the field of geophysics.[8,10,11] Two new refinements are introduced here: one based on spherical barycentric coordinates on the faces of the icosahedron and the other based on a numerical algorithm that iteratively redistributes the points to drive the D measure to zero. Finally, a new and very simple method is presented to generate a highly uniform set of points on the sphere by placing points at regular intervals on a spherical spiral. These points serve as excellent test bore sights to evaluate star identification algorithms. The uniformity of the random, icosahedral, and spiral sets are compared in the sequel using the measures just discussed.

Construction of the Voronoi Diagram

The algorithm presented here to construct the Voronoi diagram for a set of points on a sphere is based on the algorithm presented by of Guibas and Stolfi.[12] It uses their quad-edge data structure to construct the Delaunay triangulation. The quad-edge data structure simultaneously represents a subdivision of a two manifold and its dual. Thus, whereas the algorithm explicitly constructs the Delaunay triangulation, the Voronoi diagram is implicitly constructed.

Point Insertion Process

The algorithm is incremental and starts by building a scaffold from vertices of a regular spherical tetrahedron. The scaffold forms an initial Delaunay triangulation to start the algorithm and is removed at the end. Points in the set to be triangulated are inserted one by one, adding and deleting edges as necessary to maintain the Delaunay property. Figure 2 shows the insertion process,[12] which is described as follows. Locate the containing triangle in which the new point resides (bold edges in Fig. 2a). Add edges to the triangulation that connect the new point to the vertices of the containing triangle (bold edges in Fig. 2b). Hold the edges of the containing triangle suspect (dashed line bold edges in Fig. 2b). Interrogate a suspect (dashed line bold edge in Fig. 2c) by investigating whether the interrogation cap (bold circle in Fig. 2c), defined by the suspect and the new point, contains any other point in the triangulation. If not then the suspect is Delaunay and is kept in the triangulation (bold edge in Fig. 2d). The next suspect is interrogated (dashed line bold edge in Fig. 2d). If the interrogation cap contains another point of the triangulation, called a witness (bold point in Fig. 2d), then the suspect is not Delaunay and is deleted. Add an edge to the triangulation connecting the new point to the witness (bold edge in Fig. 2e), and call two new suspects into question (dashed line bold edges in Fig. 2e). Continue the interrogation of suspects until all of them have been eliminated (Fig. 2f). See Ref. 12 by Guibas and Stolfi for a proof that the resulting triangulation is Delaunay.

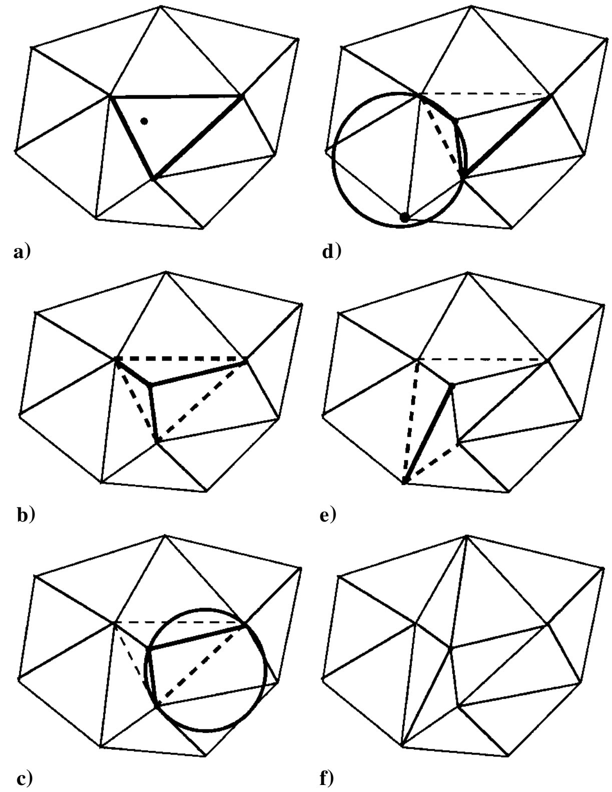


Fig. 2 Insertion of a point into a Delaunay triangulation.

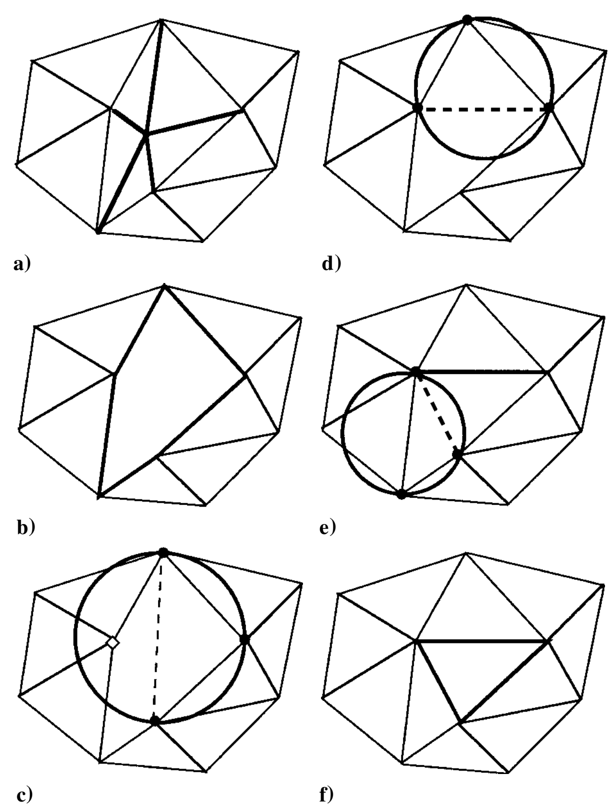


Fig. 3 Removal of a point from a Delaunay triangulation.

Point Removal Process

After all points in the set to be triangulated have been inserted, the scaffold points are removed. Figure 3 shows the removal process, which is described as follows. Remove the desired point and its connecting edges (bold edges in Fig. 3a) leaving a cavity in the triangulation (bold edges in Fig. 3b). Mend the cavity incrementally as follows. Form a nominee for the triangulation from three consecutive points on the cavity (bold points in Fig. 3c). Examine the other points on the cavity to see if any reside inside the cap circumscribing the nominee (bold circle in Fig. 3c). If so (diamond in Fig. 3c), the nominee is not Delaunay. Form a new nominee from the next three consecutive points (bold points in Fig. 3d). If no other point resides inside the cap circumscribing the nominee, the nominee is Delaunay. Add a spanning edge to include the nominee in the triangulation (bold edge in Fig. 3e), at the same time reducing the cavity. Continue checking nominees formed by consecutive points on the remaining cavity (Fig. 3e) until the cavity is reduced to a triangle (Fig. 3f).

The triangulation resulting from the point removal process just described is Delaunay, as proven by the following argument. An edge is said to be Delaunay if a circumscribing circle contains no points in its interior. A triangulation is Delaunay if and only if each of its edges is Delaunay (see Ref. 12). The edges created to mend the cavity left by the point removal are Delaunay by construction. The other edges are Delaunay because the original triangulation is Delaunay. Therefore, the triangulation resulting from the point removal process is Delaunay.

Implementation

Lischinski[13] gives a C++ implementation of the algorithm to construct the Delaunay triangulation. Although his implementation is for a set of points in the Euclidean plane, the topological aspects of the algorithm apply to points on a sphere. The necessary spherical geometry equivalents for the planar geometry functions used by Lischinski are given subsequently. The reader is referred to Lischinski for the topological functions and algorithm structure. In the sequel unit vectors are used to represent points on the sphere.

Center(A, B, C) returns the center of the spherical cap defined by the points A, B, and C. With $G = (B - A) \times (C - A)$, Center $= G/|G|$.

InCap(A, B, C, D) determines whether the point D is inside the spherical cap defined by A, B, and C. This function replaces the function InCircle used by Lischinski.[13] With $G =$ Center(A, B, C), InCap $= (G \cdot A) < (G \cdot D)$.

Proper(A, B, C) determines whether the spherical triangle defined by the points A, B, and C is proper, meaning that the triangle formed by the points taken in counterclockwise order as viewed from outside the sphere has area less than 2π. This function replaces the function (counter clockwise) used by Lischinski.[13] With $G =$ Center(A, B, C), Proper $= G \cdot A > 0$.

OnEdge(X, e) determines if the point X is within ε of the edge e. With $G =$ one endpoint of e, $H =$ the other endpoint of e, and $O =$ the sphere center, OnEdge $= |X - G| \leq |H - G|$ and $|H - X| \leq |H - G|$ and X within ε of the plane defined by G, H, and O.

The final triangulation contains $3n - 6$ edges and $2n - 4$ triangles,[4] where n is the number of points in the set. The Voronoi vertex associated with a Delaunay triangle (by duality) is defined geometrically by the center of the spherical cap circumscribing the triangle.[4] The Voronoi cell area for a point P is found by summing the area of the constituent triangles formed by P and the edges of the cell. The area of the spherical cap circumscribing a Delaunay triangle is used in the sequel to define a measure of uniformity. The area of a spherical triangle and cap are found by the following functions.[14]

TriArea(A, B, C) returns the area of the spherical triangle formed by the points A, B, and C. With $a = a\sin(|B - C|/2)$, $b = a\sin(|C - A|/2)$, $c = a\sin(|A - B|/2)$, $s = (a + b + c)/2$, TriArea $= 4a\tan\{\mathrm{sqrt}[\tan(s)\tan(s - a)\tan(s - b)\tan(s - c)]\}$.

CapArea(A, B, C) returns the area of the spherical cap defined by the points A, B, and C. With $G =$ Center(A, B, C), CapArea $= \pi|A - G|^2$.

Measures of Star Distribution

Two star distribution measures are introduced here. The D measure represents the star density variance and is based on the area of the Voronoi cells. The G measure is based on the area of the spherical caps circumscribing the Delaunay triangles. It represents how close the Delaunay triangles are to being equilateral.

Recall that the Voronoi cell for a star P consists of the region of the celestial sphere closer to P than to any other star. The inverse of the cell area, called the Voronoi density, provides an intrinsic measure of star density in the immediate neighborhood of P. The statistical variance v of the density over the celestial sphere is given by

$$v = \sum_{k=1}^{n} u_k (d_k - \bar{d})^2 \tag{1}$$

where n is the number of stars in the set, $d_k = 1/a_k$ is the Voronoi density in the kth cell, a_k is the area of the kth cell, $\bar{d} = n/(4\pi)$ is the average density over the sphere, and $u_k = a_k/(4\pi)$ is the fraction of the sphere with density d_k. If all stars have the same Voronoi cell area, v will be zero. For a uniformly random distribution of stars, the expected value of v is $\bar{d}^2/\sqrt{(2\pi)}$ as found by numerical experimentation. With this in mind, the D measure is defined as the following normalized density variance:

$$D = v\sqrt{2\pi}/\bar{d}^2 \tag{2}$$

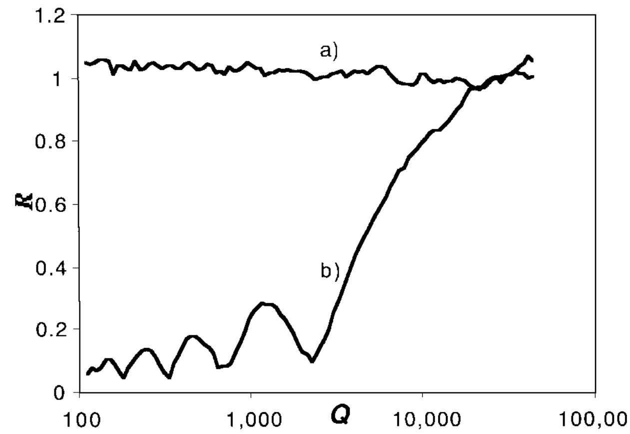
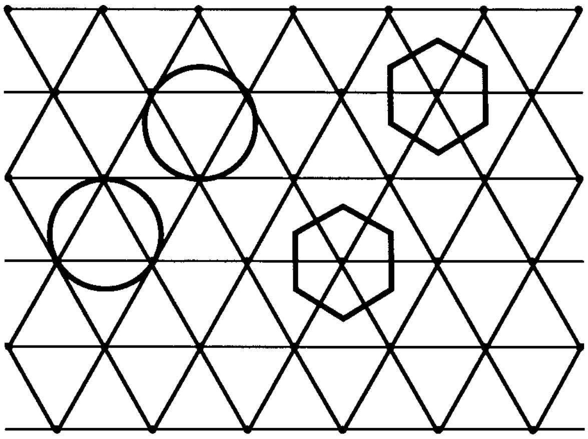


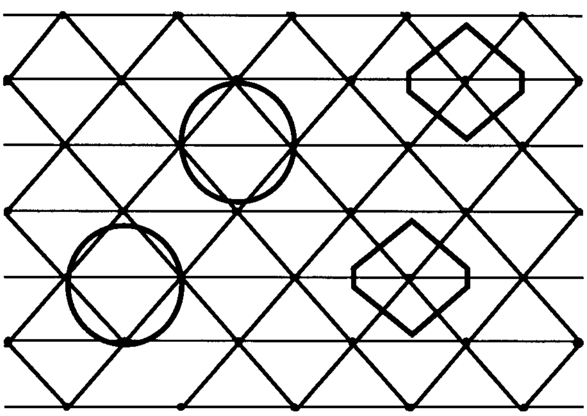
Fig. 4 Vedder R measure vs Q for 2252 randomly distributed points, a, and 2252 points distributed uniformly, b, across the faces of a spherical icosahedron.

Vedder[1] defines a similar measure of uniformity as follows. The celestial sphere is partitioned into Q nonoverlapping, contiguous cells of equal area. Then the variance of the number of stars in each cell is found. The measure $R(Q)$ is defined as the ratio of this variance to the variance of a uniformly random distribution. Vedder's R measure and the D measure defined in Eq. (2) are similar in that both represent a normalized variance of the star density, both equal one for a uniformly random distribution, and both are smaller for more uniform distributions. They differ in that the D measure is intrinsic, whereas the R measure depends on the auxiliary parameter Q and the arbitrary placement and shape of the Q regions. Figure 4 shows the R measure over a range of Q for two distributions: one random and the other highly uniform. When the distribution is random, the R measure is nearly unity regardless of the value of Q. However, when the distribution is uniform, $R(Q)$ shows a discernible variation vs Q. If one is interested in a specific value of Q, for example, $Q = 4\pi/a_{\mathrm{FOV}}$, where a_{FOV} is the area of a star sensor field of view, the variability of the $R(Q)$ may be of little consequence. In fact, the R measure may be appropriate because it reports the variance in the number of stars the star sensor will actually see. However, the R measure will indicate some nonuniformity in a distribution even if by all other accounts it is highly uniform. This feature is a result of the quantization inherent in counting the number of stars, always an integer, in each of the Q regions. Furthermore, $R(Q)$ approaches 1 as Q becomes increasingly large. The D measure, on the other hand, is an intrinsic measure of the variation in the density of stars over the sphere. If every Voronoi cell is equal in area, clearly a highly uniform condition, the D measure will report this uniformity faithfully by taking on a value of zero.

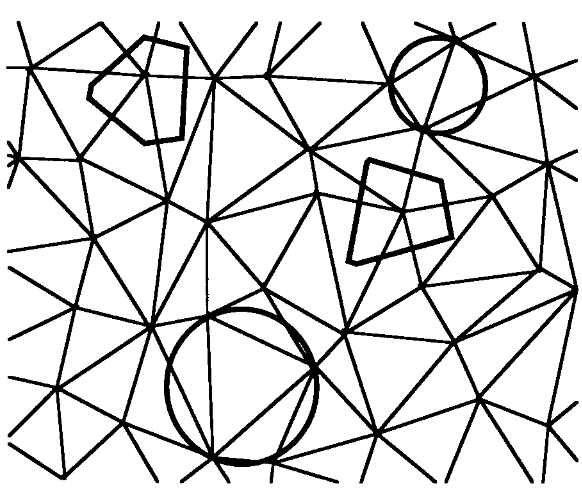
For attitude determination, one is concerned not only with the density of stars in the catalog, but also with the size of the holes between stars. The need for a second measure of uniformity is seen in Fig. 5, which shows several arrangements of points in the Euclidean plane. The Voronoi cell area is the same for every point in each arrangement. Thus, the D measures are all zero. In Fig. 5a the



a) $D = 0$ and $G = 0$



b) $D = 0$ and $G = 0.35$



c) $D = 0$ and $G = 0.65$

Fig. 5 Comparison of several sets of points, each having a D measure of zero.

points are located at the vertices of equilateral triangles. The arrangement in Fig. 5b has points located at the vertices of isosceles triangles. Whereas the arrangement appears to be very uniform, it has larger circular regions devoid of points than does the arrangement in Fig. 5a. Finally, Fig. 5c shows a relatively random arrangement of points, although one for which the Voronoi cell areas are all equal. Compared with Figs. 5a and 5b, there are also much larger regions devoid of points. The D measure alone indicates that the distribution is very uniform. A measure is needed that indicates how large are the regions devoid of points.

Such a measure will be defined. First it is necessary to examine the quintessential feature of a Delaunay triangulation and how it relates to the star catalogs. A desirable feature for a star catalog is that the area of the holes between stars be as small as possible for a given catalog size. As already discussed, the Delaunay triangulation of a set of stars has the feature that every spherical cap circumscribing a triangle in the triangulation contains no stars in its interior. Statistics on the area of these Delaunay caps provide rigorous measures of the how large are the circular holes between stars. The Delaunay triangulation of n points on the unit sphere comprises $2n - 4$ triangles.[4] The total area of the respective $2n - 4$ Delaunay caps divided by the minimum total area possible for a distribution of n points is a natural measure of the holes between stars. The problem with such a definition of uniformity is finding a general expression for the minimum total cap area for n points. An idealized minimum is used instead. The average area of the triangles is $4\pi/(2n - 4)$. The cap circumscribing a triangle of area α is smallest when that triangle is equilateral. Whereas there are only three distributions of points that result in the Delaunay triangles being equilateral (the tetrahedron, octahedron, and icosahedron), it is convenient to assume that the n points in a distribution can be arranged at the vertices of $2n - 4$ equilateral and equal area triangles. Let β be the area of a spherical cap circumscribing an equilateral spherical triangle of area $4\pi/(2n - 4)$. The G measure is now defined as follows:

$$G = 2\ln\left(\frac{1}{\beta(2n-4)}\sum_{j=1}^{2n-4} b_j\right) \tag{3}$$

where b_j is the area of the spherical cap circumscribing the jth Delaunay triangle. The application of the logarithm and scaling by two results in a measure that is zero for an ideally uniform distribution and that has an expected value of one for an uniformly random distribution. Finding the value of β to be used in Eq. (3) involves tractable, but rather messy, spherical trigonometry. For large n, area $\beta \approx \beta_\infty = 16\pi^2/[(2n - 4)\sqrt{27}]$, which is the value of β for a planar triangle of area $4\pi/(2n - 4)$.

When applied to the arrangements in Fig. 5, the G measure indicates that the arrangement in Fig. 5c is much less uniform than in Figs. 5a or 5b in agreement with one's own intuition. Note that the large regions devoid of points in Fig. 5c are not due to low point density per se (the density is uniform as indicated by the D measure) but because many of the Delaunay triangles are far from equilateral.

Construction of Uniformly Distributed Points

Before addressing the most significant topic of the paper, namely, the Voronoi density reduction method to select stars for an onboard catalog, the construction of uniformly distributed points on the sphere is considered. The ability to construct uniformly distributed points is important for attitude determination. Such points serve as star sensor bore sight test cases when evaluating the performance of a stellar attitude determination algorithm and are needed for some star catalog generation methods.[3,9] Several methods to construct n uniformly distributed points are prescribed next. Uniformity measures for the resulting point sets are reported in Table 1.

To begin, a prescription is given for a uniformly random distribution.[1,7] Let r_1 be a uniformly distributed random number on the interval $[-1, +1]$ and r_2 be a second on the interval $[0, 2\pi)$. Then the coordinates of a random point on the sphere are given by

$$x = \cos(r_2)\sqrt{1 - r_1^2} \tag{4}$$

$$y = \sin(r_2)\sqrt{1 - r_1^2} \tag{5}$$

$$z = r_1 \tag{6}$$

Choosing n points according to Eqs. (4–6) yields a uniformly random distribution over the unit sphere. By uniformly random it is meant that the probability of having a point in any very small region of area a is $na/(4\pi)$ and is independent of the placement of any other point in the set. The D, G, and R measures of such a set are all very nearly 1 (Table 1).

Williamson[8] and Ketchum and Tolson[9] have suggested a method to distribute points with icosahedral symmetry. Points are placed at regular intervals on the faces of an icosahedron, and then projected radially onto the unit sphere. Algorithm 1 shows an implementation of this approach.

Table 1 Comparison of several sets of n = 2562 points

Set	D	G	R[a]	$E_0 - f_0$[b]	$E_{-1} - f_{-1}$[c]	VCA $\cdot n/4\pi$[d] Min.	Max.	DCA/β[e] Min.	Max.	DEL/γ[f] Min.	Max.
Random	1.0830	0.9886	1.0417	5049.73	71135.90	0.0566	3.8239	0.0111	8.4429	0.0141	3.3396
Simple icosahedral	0.0484	0.0089	0.1637	653.075	4254.47	0.6193	1.1976	0.6765	1.0976	0.7684	1.0976
Barycentric icosahedral	0.0041	0.0214	0.1054	−9.26153	316.889	0.8362	1.056	0.9134	1.0642	0.8929	1.0606
Homogenized icosahedral	0	0.0349	0.1026	−67.2734	11.234	1	1	0.9878	1.0892	0.8981	1.1455
Baumgardner and Frederickson[11] icosahedral	0.0135	0.0351	0.1201	81.3457	1009.59	0.8863	1.1950	0.9587	1.2054	0.9193	1.0976
Williamson[8] icosahedral	0.0135	0.0320	0.1228	68.6994	943.086	0.8863	1.1863	0.8908	1.2079	0.9042	1.1405
Vestine et al.[10] icosahedral	0.0049	0.0318	0.1202	46.5965	523.088	0.8863	1.0850	0.9562	1.1385	0.9193	1.1247
Bauer spiral	5.60e−5	0.1896	0.1118	−62.5544	145.18	0.9553	1.0472	0.8850	1.3207	0.8222	1.3256
Refs. 5 and 6 spiral	3.15e−4	0.1944	0.1167	−61.5642	157.109	0.7252	1.0627	0.4911	1.4159	0.5251	1.3684

[a] R given for $Q = 817$. [b] Logarithmic potential energy E_0 is offset by $f_0 = -638988.52816$ (see Ref. 6).
[c] Coulomb potential energy E_{-1} is offset by $f_{-1} = 3210303.93486$ (see Ref. 6). [d] Normalized by dividing by $4\pi/n$.
[e] Normalized by dividing by $\beta = 16\pi^2/[(2n-4)\sqrt{27}]$, which for large n approximates the area of a cap circumscribing an equilateral triangle of area $4\pi/(2n-4)$.
[f] Delaunay edge length (DEL) is normalized by dividing by $\gamma = \{16\pi/[(2n-4)\sqrt{3}]\}^{1/2}$, which for large n approximates the edge length of an equilateral triangle of area $4\pi/(2n-4)$.

Algorithm 1: Construction of Points with Icosahedral Symmetry

1) Round n, the minimum number of points to be constructed, up to a number that can have icosahedral symmetry,

$$m = \text{ceiling}\{\text{sqrt}[(n-2)/10]\}, \qquad n = 10m^2 + 2$$

2) Find the four vertices of two adjacent faces of the icosahedron

$$\gamma = 2\arctan\left[\left(\sqrt{5}-1\right)/2\right]$$

$$\alpha = [\cos(\pi/5)\sin(\gamma), -\sin(\pi/5)\sin(\gamma), \cos(\gamma)]$$

$$\beta = [\cos(\pi/5)\sin(\gamma), +\sin(\pi/5)\sin(\gamma), \cos(\gamma)]$$

$$\chi = (0, 0, 1), \qquad \delta = [0, 0, -\cos(\gamma)]$$

3) Construct a set Ω_1 of m^2 points on the two faces accordingly. For $i = 0$–$(m-1)$ and $j = 1$–$m, \ldots,$ a) find the point index k and face coordinates (h_a, h_b, h_c)

$$k = i \cdot m + j, \qquad h_a = i \cdot m^{-1}$$

$$h_b = j \cdot m^{-1}, \qquad h_c = 1 - (h_a + h_b)$$

b) resolve the face coordinates so that $h_a, h_b, h_c \geq 0$; if $h_c \geq 0$

$$\boldsymbol{A} = \alpha, \qquad \boldsymbol{B} = \beta, \qquad \boldsymbol{C} = \chi$$

otherwise

$$\boldsymbol{A} = \beta, \qquad \boldsymbol{B} = \alpha, \qquad \boldsymbol{C} = \delta$$

$$h_a = 1 - h_a, \qquad h_b = 1 - h_b, \qquad h_c = -h_c$$

and c) convert (h_a, h_b, h_c) to a unit vector:

$$\boldsymbol{P} = h_a\boldsymbol{A} + h_b\boldsymbol{B} + h_c\boldsymbol{C}, \qquad \boldsymbol{P}_k = \boldsymbol{P}/|\boldsymbol{P}|$$

4) Construct four sets of points Ω_2–Ω_5 by duplicating each point in Ω_1 and rotating about the Z axis by an angle $\theta_r = 2\pi(r-1)/5$, $r = 2$–5.

5) Let Ω_U be the union of points in Ω_1–Ω_5 and the point (0, 0, +1), a total of $5m^2 + 1$ points. Construct the set of points Ω_L by reflecting each point in Ω_U about the XY plane. The final set of points Ω is the union of points in Ω_L and Ω_U, a total of $10m^2 + 2$ points.

Whereas the simple icosahedral set constructed by Algorithm 1 is much more uniform than the random set and is relatively easy to generate, the point density varies by almost a factor of two over the sphere. (See minimum and maximum Voronoi cell area reported in Table 1.) A refinement is made by interpreting the face coordinates (h_a, h_b, h_c) in step 3c as barycentric coordinates on the sphere. In the Euclidean plane a point $\boldsymbol{P}$ inside the triangle $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ can be represented by

$$\boldsymbol{P} = h_a\boldsymbol{A} + h_b\boldsymbol{B} + h_c\boldsymbol{C} \tag{7}$$

where (h_a, h_b, h_c) are the barycentric coordinates[15] and equal the following ratios of triangle areas:

$$h_a = \text{Area}(\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C})/\text{Area}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \tag{8}$$

$$h_b = \text{Area}(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{C})/\text{Area}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \tag{9}$$

$$h_c = \text{Area}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{P})/\text{Area}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \tag{10}$$

The barycentric coordinates may be generalized to spherical triangles by interpreting the Area function in Eqs. (8–10) as the area of the spherical triangle. (See TriArea defined earlier.) Using this interpretation, the spherical barycentric coordinates (h_a, h_b, h_c) uniquely define the point $\boldsymbol{P}$ on the spherical triangle $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$. Algorithm 2 uses an iterative technique to solve for $\boldsymbol{P}$ in Eqs. (8–10).

Algorithm 2: Conversion of Spherical Barycentric Coordinates to a Unit Vector

1) Inputs are the triangle vertices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ and the barycentric coordinates (h_a, h_b, h_c).

2) Set $\mu = \text{TriArea}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$.

3) Initialize the triplet (t_a, t_b, t_c) and the point $\boldsymbol{P}$ accordingly:

$$t_a = h_a, \qquad t_b = h_b, \qquad t_c = h_c$$

$$\boldsymbol{P} = h_a\boldsymbol{A} + h_b\boldsymbol{B} + h_c\boldsymbol{C}, \qquad \boldsymbol{P} = \boldsymbol{P}/|\boldsymbol{P}|$$

4) Iterate the following equations at least three times:

$$t_a = t_a + h_a - \text{TriArea}(\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C})/\mu$$

$$t_b = t_b + h_b - \text{TriArea}(\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{C})/\mu$$

$$t_c = t_c + h_c - \text{TriArea}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{P})/\mu$$

$$\boldsymbol{P} = t_a\boldsymbol{A} + t_b\boldsymbol{B} + t_c\boldsymbol{C}, \qquad \boldsymbol{P} = \boldsymbol{P}/|\boldsymbol{P}|$$

5) Return $\boldsymbol{P}$.

Replacing step 3c in Algorithm 1 with Algorithm 2 yields a very uniform set of points called the barycentric icosahedral set. Its point density varies by only a factor of 1.26 over the sphere, and the largest Voronoi cell is only 5.6% larger than the average (Table 1).

The icosahedral set can be made more uniform by homogenizing its distribution. To do so, the points are nudged from regions of higher density toward regions of lower density using Algorithm 3 until all points have the same Voronoi cell area.

Algorithm 3: Homogenize the *n* Points in the Set Ω

1) Build the Voronoi diagram for the set Ω.

2) Determine the Voronoi cell area (VCA) gradient $\boldsymbol{x}_k$ for each point $\boldsymbol{P}_k$ in Ω:

$$x_k = \frac{1}{3}\sum_{j \in J_k}(a_j - a_k)(\boldsymbol{P}_j - \boldsymbol{P}_k)/|\boldsymbol{P}_j - \boldsymbol{P}_k|^2$$

where $\boldsymbol{J}_k$ is the set of indices for the nearest neighbors of $\boldsymbol{P}_k$ and a_k is the area of the kth Voronoi cell.

3) Nudge each point $\boldsymbol{P}_k$ and normalize as follows:

$$\boldsymbol{P}_k = \boldsymbol{P}_k + c\boldsymbol{x}_k, \qquad \boldsymbol{P}_k = \boldsymbol{P}_k/|\boldsymbol{P}_k|$$

where $c = 0.75$ is a convergence parameter.

4) Rebuild the Voronoi diagram using the nudged points from step 3. Evaluate the D measure. If necessary repeat steps 2–4.

Algorithm 1 followed by Algorithm 3 yields a very uniform and highly symmetric set of points. The drawback is that it is a very complicated algorithm. Other refinements of the simple icosahedral set are possible. Table 1 reports on the uniformity for point sets constructed according to Williamson,[8] Vestine et al.,[10] and Baumgardner and Frederickson.[11] None of these point sets is as uniform as either the barycentric or homogenized sets.

Finally, a spiral set is defined that is very uniform but much easier to construct than the icosahedral sets. It does, however, have larger holes between neighboring points as indicated by the G measure. Rakhmanov et al.[5] (see also Ref. 6) give a similar spiral but one that this author found to be slightly less uniform (Table 1).

A spherical spiral $\boldsymbol{C}$ is described in spherical coordinates by the following equations:

$$r = 1, \qquad \theta = L\phi, \qquad 0 \le \phi \le \pi \tag{11}$$

where L specifies the spiral slope. One can see that when θ increases by 2π, that is, one turn of the spiral, ϕ increases by $2\pi/L$. Thus, the spacing σ between turns of the spiral is given by

$$\sigma = 2\pi/L \tag{12}$$

The total arc length s_π of the spiral is given by the following integral[16]:

$$S_\pi = \int_C \sqrt{d\phi^2 + d\theta^2 \sin^2\phi} = \int_0^\pi \sqrt{1 + L^2\sin^2\phi}\, d\phi$$
$$= 2\sqrt{1+L^2}\, E\left(L/\sqrt{1+L^2}\right) \tag{13}$$

where $E(\,)$ is the complete elliptic integral of the second kind. To construct the spiral point set, the spiral is divided into n nearly equal length segments, and a point is place at the center of each segment. The spiral slope L is chosen so that the spacing between turns equals the length of the segments. Setting $\sigma = s_\pi/n$, using the approximation $s_\pi \approx 2L$ that holds for large L, and substituting $\sigma = 2\pi/L$, we find the desired slope to be

$$L = \sqrt{n\pi} \tag{14}$$

Now, the arc length s from $(\phi, \theta) = (0, 0)$ to $(\phi, L\phi)$ is approximately equal to

$$s \approx L(1 - \cos\phi) \tag{15}$$

as can be seen by considering an indefinite integral in Eq. (13) and making the assumption that $L^2 \sin^2\phi \gg 1$. By the use of Eqs. (11), (14), and (15), Algorithm 4 places n points at nearly equal intervals along the spiral.

Algorithm 4: Construction of Points on a Spherical Spiral

1) Set $L = \sqrt{(n\pi)}$, where n is the number of points to be constructed.

2) For $k = 1$–n, a) find the spherical coordinates of the kth point,

$$z_k = 1 - (2k-1)/n, \qquad \phi_k = \arccos(z_k), \qquad \theta_k = L\phi_k$$

and b) convert to rectilinear coordinates

$$x_k = \sin(\phi_k)\cos(\theta_k), \qquad y_k = \sin(\phi_k)\sin(\theta_k)$$

$$\boldsymbol{P}_k = (x_k, y_k, z_k)$$

Table 1 compares the several point sets suggested here. The spiral set is very easy to construct and is remarkably uniform. It makes an excellent set of points to serve as star sensor bore sight test cases when evaluating the performance of a stellar attitude determination algorithm. If one needs a highly uniform distribution, the homogenized icosahedral set is best, but requires a very complicated algorithm to generate. The barycentric icosahedral set is nearly as uniform as the homogenized set, but is much easier to generate.

Star Catalog Generation

A new method is now presented to select stars for an onboard catalog from a larger candidate set based on the Voronoi diagram of the candidates. Often it is the case that a star sensor can reliably sense many more stars than needed for onboard attitude determination. For example, a sensor with an 8-deg (diameter) circular field of view may reliably sense 2000 stars, whereas 817 stars (about one per sensor field of view) may be enough to maintain a good attitude estimate if the stars are uniformly distributed. Selecting a uniform set of stars will tend to minimize regions devoid of stars. The selection problem is formally stated as follows: Given a candidate set of m stars that meet specific mission and sensor criteria, select a subset of n stars for an onboard catalog that is as uniform as possible. Intuitively, it would seem that if one could first identify the densest region in the candidate set, and then remove a star from that region, the resulting reduced set would be more uniform. Repeating this "greedy" process until the desired number of stars remains should result in an onboard catalog that is very uniform.

The Voronoi diagram of the candidate set provides an excellent framework to identify the densest region. As mentioned earlier, the inverse of the VCA may be interpreted as the star density within the respective cell. The candidate with the smallest VCA is then a star in the densest region of the candidate set. Using this definition of star density, the Voronoi density reduction (VDR) method is devised to select stars for an onboard catalog from a larger candidate set. First, the Voronoi diagram of the candidate set is built. Next, the star with the smallest Voronoi cell area is removed, and the Voronoi diagram of the reduced candidate set is found. This removal process is repeated until the desired number of stars remains.

Of course, the question is: Does the VDR method result in the most uniform onboard catalog possible, that is, is it optimal? This question is a difficult one considering that choosing a subset of n stars from a larger set of m stars can be done in $m!/[(m-n)!\,n!]$ ways, which is a very large number for typical values of m and n. The optimality of the VDR method is not addressed here. However, the VDR method is demonstrated to be superior to several existing methods by three representative examples.

Random Candidates

The first example demonstrates the basic VDR method and compares it to other methods to selects stars from a candidate set. The problem is stated as follows: given 2000 randomly distributed candidate stars select 817 stars that are uniformly distributed, for an onboard catalog. The VDR method is applied to this problem. The resulting onboard catalog is much more uniform than those chosen by other known methods[1−3] (Table 2). The other methods are described next.

Barry et al.[2] suggest a method that has the same structure as the VDR method but uses a different measure of density. For each star in the candidate set, the sum of the angular separations to the three nearest stars is found. The star with the smallest sum of separations is removed to form a reduced candidate set. This removal process is repeated until the desired number of stars remains. Although this method yields very satisfactory results, it is not quite as good as the VDR method (Table 2).

Vedder[1] presents a method in which the sky is divided into contiguous and equal area cells. The cells are congruent and nearly

Table 2 Comparison of onboard catalogs generated by several methods

Selection method	No. of stars	Measure of uniformity		Delaunay cap radius, rad	
		D	G	Min.	Max.
Candidates	2000	1.0251	0.9911	0.0094	0.1426
VDR	817	0.0570	0.5061	0.0450	0.1426
Barry et al.[2]	817	0.0753	0.5015	0.0529	0.1426
Vedder[1]	815	0.2218	0.7878	0.0290	0.2618
Uniform ref.	818	0.1400	0.5754	0.0289	0.1426

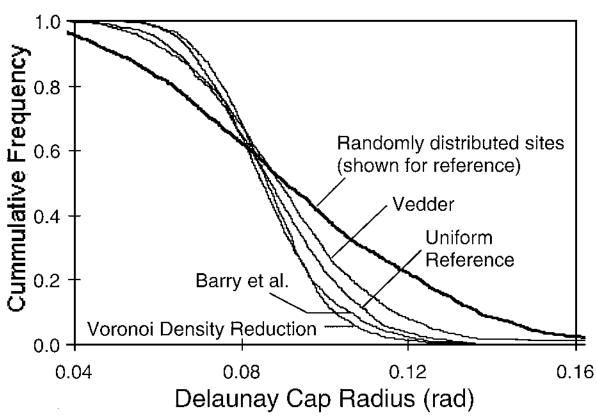


Fig. 6 Cumulative frequency of Delaunay caps with radii greater than the abscissa for onboard catalogs selected by several methods (see Table 2).

square in cylindrical coordinate space. One candidate star from each cell is selected for the onboard catalog. Here the star closest to the center of the cell is selected. Because some cells will contain no candidate, the number of cells must be somewhat greater than the number of stars desired in the onboard catalog. Choosing the number of cells to yield exactly the desired number of onboard stars is difficult because of the arbitrary placement of cells. After a reasonable amount of trial and error, 815 stars were chosen using 935 cells (55 × 17). Some heuristic method, not addressed here, must be applied to select two additional stars. The set of 815 stars, although more uniform than the candidate set, is much less uniform that the set generated using the VDR method (Table 2).

Considered last is the uniform reference method, a refinement of a method suggested by Kudva.[3] A set of reference points is uniformly distributed across the celestial sphere. The candidate star closest to each reference point is selected for the onboard catalog. The spiral set constructed by Algorithm 4 is used for the reference points. Because reference points in sparse regions of the candidate set will tend to share the same closest candidate, the number of reference points must be somewhat larger than the desired number of onboard stars. As with the Vedder method,[1] choosing the number of reference points to yield exactly the desired number of onboard stars proves to be difficult. Using 837 reference points, 818 onboard stars were selected. Again, the resulting set of onboard stars is not as uniform as the set found by the VDR method (Table 2).

To gain a better appreciation for the uniformity of the onboard catalogs generated by the methods just discussed, the cumulative frequency[17] (CF) of Delaunay cap radii is examined. Recall that a Delaunay cap, which circumscribes a triangle in the Delaunay triangulation, contains no other stars in its interior. Thus, the Delaunay caps represent the empty regions between stars. Figure 6 shows the CF of Delaunay cap radii for the onboard catalogs. For comparison the CF for a randomly selected set of 817 stars is also shown. The catalog generated by the VDR method has the smallest CF at larger values of the Delaunay cap radius, indicating that it contains the fewest large regions devoid of stars.

The common way to examine the distribution of stars is to find the statistics of having k stars in the sensor field of view. For the present example the field of view is assumed to be 8 deg (diameter) circular. To accumulate the statistics, 16,000 sample sensor bore sights are generated using the spiral method of Algorithm 4. For

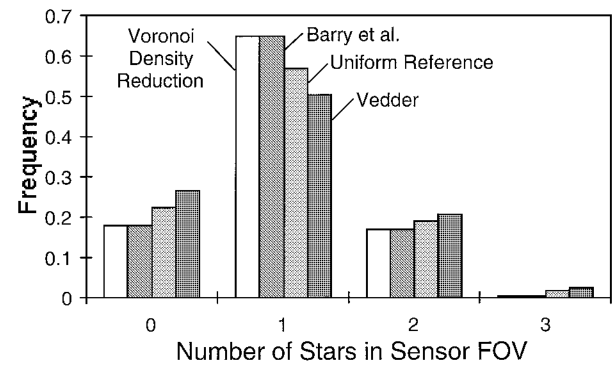


Fig. 7 Frequency of k = 0, 1, 2, and 3 stars in a sensor field of view for onboard catalogs selected by several methods (see Table 2).

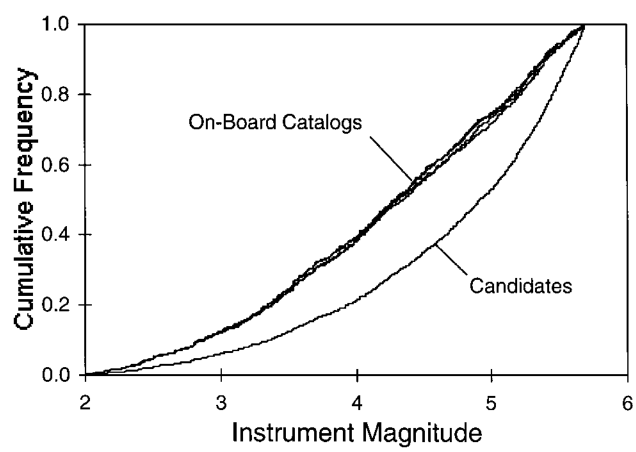


Fig. 8 CF of stars whose magnitude is greater than the abscissa for EOS-AM1 onboard catalogs selected by several methods that favor brighter stars (see Table 3).

each bore sight, the number of catalog stars within in 4 deg of the bore sight is determined, and statistics on there being k stars, $k = 0$–3, are accumulated (Fig. 7). The VDR and Barry et al.[2] methods result in the smallest frequency of no stars ($k = 0$) and the highest frequency of one star ($k = 1$) in the sensor field of view, attesting to their efficacy in generating onboard catalogs.

It would seem, according to Fig. 7, that the VDR and Barry et al.[2] methods are equivalent: the frequency of k stars, $k = 0$–3, in the sensor field of view is the same for both. However, the CF of DCAs (Fig. 6) indicates that the regions with no stars in the sensor field of view are more dispersed using the VDR method than for the Barry et al.[2] method. In other words, whereas both methods yield the same fraction of the sky with no stars in the sensor field of view, the VDR method has smaller contiguous regions with no stars.

Earth Observing Satellite (EOS)-AM1 Catalog

The Earth observing satellite (EOS)-AM1 spacecraft is limited to 700 stars in its onboard catalog. There are, however, 1523 candidate stars that satisfy the criteria for the mission including magnitude range, near neighbor magnitude vs angular separation, proper motion, position errors, and magnitude errors. The baseline onboard star catalog for EOS-AM1 was chosen with a bias toward brighter stars.[3,18,19] The selection methods described earlier are easily modified to favor brighter stars. The VDR method is modified to remove the star with the smallest intensity weighted VCA w_k during each iteration.

$$w_k = a_k \exp(-g m_k) \tag{16}$$

where $g > 0$ is the weighting parameter, a_k is the VCA of the kth star, and m_k is the instrument magnitude of the kth star. With $g = 0.3$, the catalog generated by the VDR method exhibits approximately the same magnitude distribution as the baseline catalog (Fig. 8) but is more uniform (Table 3). The Barry et al.,[2] Vedder,[1] and uniform reference methods can be similarly modified to favor brighter stars.

Table 3 Comparison of onboard catalogs for EOS-AM1

Method	No. of stars	Measure of uniformity		Delaunay cap radius	
		D	*G*	Min.	Max.
Candidates	1523	1.0134	0.9975	0.0076	0.1485
VDR	700	0.1379	0.6017	0.0398	0.1485
Barry et al.[2]	700	0.1533	0.6020	0.0476	0.1626
Vedder[1]	706	0.2702	0.8097	0.0136	0.2304
Uniform ref.	699	0.2569	0.7106	0.0360	0.1675
Baseline	700	0.2615	0.6887	0.0391	0.1603

Table 4 Comparison of onboard catalogs for the Space Shuttle

Method	No. of stars	Measure of uniformity		Delaunay cap radius	
		D	*G*	Min.	Max.
Candidates	273	1.4926	0.9393	0.0243	0.3025
VDR	100	0.0821	0.4695	0.1632	0.3217
Barry et al.[2]	100	0.1074	0.4804	0.1680	0.3598
Vedder[1]	101	0.2076	0.7268	0.1184	0.4834
Uniform ref.	101	0.2261	0.6294	0.0920	0.3622
Baseline	100	0.9822	0.8602	0.0876	0.5797

None of them, however, results in a catalog that is as uniform as that resulting from the VDR method.

Space Shuttle Catalog

The Space Shuttle onboard catalog is composed of 100 stars. There are 273 candidate stars that satisfy the mission requirements.[2,19,20] Of these candidates, 50 are bright, easily recognizable stars that are sighted by astronauts with a manual optical sight. Therefore, they are selected for the permanent onboard catalog. The problem remains to choose 50 additional stars such that the resulting catalog is as uniform as possible. To solve this problem, a modification to the VDR method is made: the smallest VCA that is not one of the 50 easily recognizable stars is removed during each iteration. The Barry et al.[2] method can be similarly modified. For Vedder's method[1] if a cell contains one of the 50 easily recognizable stars, no further selection is made. Otherwise, the star closest to the center of the cell is selected for the onboard catalog as before. No modification to the uniform reference method is made, other than to form the union of stars it chooses and the 50 easily recognizable stars.

Table 4 compares the uniformity of the onboard catalogs resulting from the VDR, Barry et al.,[2] Vedder,[1] and uniform reference methods with the modifications just discussed. All four methods result in catalogs more uniform than the candidate set and more uniform than the baseline onboard catalog currently in use by the Space Shuttle. Once again, the VDR method yields the most uniform catalog.

Conclusions

A new framework has been introduced to analyze the spatial distribution of stars in a catalog, namely, the Voronoi diagram/Delaunay triangulation. The Voronoi diagram subdivides the celestial sphere into polygonal cells, one for each star, such that each cell contains that part of the sphere closer to the respective star than to any other star. The inverse of the VCA, called the Voronoi density, may be interpreted as the star density within the cell. The Delaunay triangulation is the topological dual of the Voronoi diagram with the important property that the spherical caps circumscribing triangles in the triangulation are devoid of stars. Thus, the Delaunay caps may be interpreted as the empty regions between stars. Carefully chosen statistics on the Voronoi density and DCAs provide meaningful measures of star catalog uniformity. Here, two such measures have been defined: the *D* measure that is a normalized Voronoi density variance and the *G* measure that is a measure of how close to equilateral the Delaunay triangles are. Three other measures of uniformity from the scientific and engineering literature have been discussed and compared to the *D* and *G* measures.

Several methods to generate uniformly distributed sets of points on the sphere have been presented. By examining the uniformity measures of these sets, one can make an objective assessment of the benefits of each to the task at hand vs the effort involved in their generation. The spiral set introduced here is very easy to generate and yet is remarkably uniform.

The VDR method, based on the Voronoi diagram of candidate stars, has been introduced to select stars for an onboard catalog. The star in the candidate set having the smallest VCA may be interpreted as being in the densest region of the sky. Its removal results in a reduced set that is more uniform. Repeatedly removing the star having the smallest VCA until the desired number of stars remains yields a very uniform onboard catalog. The method is easily adapted to special requirements such as favoring brighter stars and retaining a preselected subset of the candidates. This method has been shown to yield a more uniform onboard catalog for both the EOS-AM1 and Space Shuttle missions than their respective baselines.

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